LUSIN-TYPE THEOREMS FOR CHEEGER DERIVATIVES ON METRIC MEASURE SPACES

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Abstract. A theorem of Lusin states that every Borel function on $\mathbb{R}$ is equal almost everywhere to the derivative of a continuous function. This result was later generalized to $\mathbb{R}^n$ in works of Alberti and Moonens-Pfeffer. In this note, we prove direct analogs of these results on a large class of metric measure spaces, those with doubling measures and Poincaré inequalities, which admit a form of differentiation by a famous theorem of Cheeger.

1. Introduction

A classical theorem of Lusin [17] states that for every Borel function f on $\mathbb{R}$, there is a continuous function u on $\mathbb{R}$ that is differentiable almost everywhere with derivative equal to f.

In [1], Alberti gave a related result in higher dimensions. He proved the following theorem, in which $|\cdot|$ denotes Lebesgue measure and Du denotes the standard Euclidean derivative of u.

Theorem 1.1 ([1], Theorem 1)**.** *Let $\Omega \subset \mathbb{R}^k$ be open with $|\Omega| < \infty$, and let $f\colon \Omega \to \mathbb{R}^k$ be a Borel function. Then for every $\epsilon > 0$, there exist an open set $A \subset \Omega$ and a function $u \in C^1_0(\Omega)$ such that*
(a) $|A| \leq \epsilon|\Omega|$,
(b) $f = Du$ *on* $\Omega \setminus A$, *and*
(c) $\|Du\|_p \leq C\epsilon^{\frac{1}{p}-1}\|f\|_p$ *for all* $p \in [1, \infty]$.
Here $C > 0$ is a constant that depends only on k.

In other words, Alberti showed that it is possible to arbitrarily prescribe the gradient of a C^1_0 function u on $\Omega \subset \mathbb{R}^k$ off of a set of arbitrarily small measure, with quantitative control on all L^p norms of Du.

Moonens and Pfeffer [18] applied Alberti's result to show a more direct analog of the Lusin theorem in higher dimensions:

Theorem 1.2 ([18], Theorem 1.3)**.** *Let $\Omega \subset \mathbb{R}^k$ be an open set and let $f\colon \Omega \to \mathbb{R}^k$ be measurable. Then for any $\epsilon > 0$, there is an almost everywhere differentiable function $u \in C(\mathbb{R}^k)$ such that*
(a) $\|u\|_\infty \leq \epsilon$ *and* $\{u \neq 0\} \subset \Omega$,
(b) $Du = f$ *almost everywhere in* Ω, *and*
(c) $Df = 0$ *everywhere in* $\mathbb{R}^k \setminus \Omega$.

These "Lusin-type" results for derivatives in Euclidean space have applications to integral functionals on Sobolev spaces [1], to the construction of horizontal surfaces in the Heisenberg group ([2, 11]) and in the analysis of charges and normal currents [18]. In addition, we remark briefly that the results of Alberti and Moonens-Pfeffer have been generalized to higher order derivatives on Euclidean space in the work of Francos [10] and Hajłasz-Mirra [11], though we do not pursue those lines here.

The purpose of this note is to extend the results of Alberti and Moonens-Pfeffer, in a suitable sense, to a class of metric measure spaces on which differentiation is defined.

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In his seminal 1999 paper, Cheeger [6] defined (without using this name) the notion of a "measurable differentiable structure" for a metric measure space. Cheeger showed that a large class of spaces, the so-called PI spaces, possess such a structure. A differentiable structure endows a metric measure space with a notion of differentiation and a version of Rademacher's theorem: every Lipschitz function is differentiable almost everywhere with respect to the structure.

The class of PI spaces includes Euclidean spaces, all Carnot groups (such as the Heisenberg group), and a host of more exotic examples like those of [5], [16], and [7].

We prove the following two analogs of the results of Alberti and Moonens-Pfeffer for PI spaces. All the definitions are given in Section 2 below.

Theorem 1.3. *Let (X,d,μ) be a PI space and let $\{(U_j,\phi_j\colon X\to\mathbb{R}^{k_j})\}_{j\in J}$ be a measurable differentiable structure on X. Then there are constants $C,\eta>0$ with the following property:*

Let $\Omega\subset X$ be open with $\mu(\Omega)<\infty$ and let $\{f_j\colon U_j\cap\Omega\to\mathbb{R}^{k_j}\}_{j\in J}$ be a collection of Borel functions. Then for all $\epsilon>0$ there is an open set $A\subset\Omega$ and a Lipschitz function $u\in C_0(\Omega)$ such that

$$\mu(A)\leq\epsilon\mu(\Omega), \tag{1.4}$$

$$f_j = d^j u \text{ a.e. on } U_j\cap(\Omega\setminus A) \tag{1.5}$$

for all $j\in J$,

$$\|\mathrm{Lip}_u\|_p \leq C\epsilon^{\frac{1}{p}-\frac{1}{\eta}}\left(\sum_{j\in J}(\mathrm{LIP}(\phi_j))^p\int_{\Omega\cap U_j}|f_j|^p\right)^{1/p} \tag{1.6}$$

for all $p\in[1,\infty)$, and

$$\|\mathrm{Lip}_u\|_\infty \leq C\epsilon^{-\frac{1}{\eta}}\sup_{j\in J}\left(\mathrm{LIP}(\phi_j)\|f_j\|_\infty\right). \tag{1.7}$$

The constants $C,\eta>0$ depend only on the data of X.

As u is Lipschitz, the Poincaré inequality (see Lemma 2.4) will allow us to also control the global Lipschitz constant of u in Theorem 1.3, and conclude that

$$\mathrm{LIP}(u)\leq C\epsilon^{-\frac{1}{\eta}}\sup_{j\in J}((\mathrm{LIP}(\phi_j))\|f_j\|_\infty),$$

if the right-hand side is finite.

That the bounds (1.6) and (1.7) involve the the chart functions ϕ_j is in some sense inevitable, as one can easily discover by looking at the measurable differentiable structure $(\mathbb{R},\phi(x)=2x)$ on $\mathbb{R}$. Note also that, unlike in the Euclidean setting of Theorem 1.1, the notion of C^1 regularity is not defined in PI spaces. Thus, the natural regularity for our constructed function u in Theorem 1.3 is Lipschitz.

Our second result is the analog in PI spaces of Theorem 1.2:

Theorem 1.8. *Let (X,d,μ) be a PI space. Let $\{(U_j,\phi_j\colon U_j\to\mathbb{R}^{k_j})\}$ be a measurable differentiable structure on X. Let $\Omega\subset X$ be open, let $\epsilon>0$, and let $\{f_j\colon U_j\cap\Omega\to\mathbb{R}^{k_j}\}$ be a collection of Borel functions.*

Then there is a continuous function u on X that is differentiable almost everywhere and satisfies

$$\|u\|_\infty\leq\epsilon \text{ and } \{u\neq 0\}\subset\Omega, \tag{1.9}$$

$$d^j u = f_j \text{ a.e. in } U_j\cap\Omega \tag{1.10}$$

for each $j\in J$, and

$$\mathrm{Lip}_u = 0 \text{ everywhere in } X\setminus\Omega. \tag{1.11}$$

In particular, Theorems 1.3 and 1.8 allow one to prescribe the horizontal derivatives of functions on the Heisenberg group, or to prescribe the one-dimensional Cheeger derivatives of functions on the Laakso-type spaces of [16] and [7].

2. Definitions and Preliminaries

We will work with metric measure spaces (X, d, μ) such that (X, d) is complete and μ is a Borel regular measure. If the metric and measure are understood, we will denote such a space simply by X. An open ball in X with center x and radius r is denoted $B(x, r)$. If $B = B(x, r)$ is a ball in X and $\lambda > 0$, we write $\lambda B = B(x, \lambda r)$.

We generally use C and C' to denote positive constants that depend only on the quantitative data associated to the space X (see below); their values may change throughout the paper.

If $\Omega \subset X$ is open, we let $C_c(\Omega)$ denote the space of continuous functions with compact support in Ω. We also let $C_0(\Omega)$ denote the completion of $C_c(\Omega)$ in the supremum norm. Any function in $C_0(\Omega)$ admits a natural extension by zero to a continuous function on all of X.

Recall that a real-valued function u on a metric space (X, d) is *Lipschitz* if there is a constant $L \geq 0$ such that

$$|u(x) - u(y)| \leq Ld(x, y) \text{ for all } x, y \in X.$$

The infimum of all $L \geq 0$ such that the above inequality holds is called the *Lipschitz constant* of u and is denoted $\mathrm{LIP}(u)$.

Given a real-valued (not necessarily Lipschitz) function u on X, we also define its pointwise upper Lipschitz constant at points $x \in X$ by

$$\mathrm{Lip}_u(x) = \limsup_{r \to 0} \frac{1}{r} \sup_{d(x,y)<r} |u(y) - u(x)|.$$

Two basic facts about Lip are easy to verify. First, for any two functions f and g,

$$\mathrm{Lip}_{f+g}(x) \leq \mathrm{Lip}_f(x) + \mathrm{Lip}_g(x). \tag{2.1}$$

Second, if f and g are Lipschitz functions, then

$$\mathrm{Lip}_{fg}(x) \leq f(x)(\mathrm{Lip}_g(x)) + g(x)(\mathrm{Lip}_f(x)). \tag{2.2}$$

A non-trivial Borel regular measure μ on a metric space (X, d) is a *doubling measure* if there is a constant $C > 0$ such that $\mu(B(x, 2r) \leq C\mu(B(x, r))$ for every ball $B(x, r)$ in X. The existence of a doubling measure μ on (X, d) implies that (X, d) is a *doubling metric space*, i.e. that every ball can be covered by at most N balls of half the radius, for some fixed constant N. In particular, a complete metric space with a doubling measure is *proper*: every closed, bounded subset is compact.

Definition 2.3. A metric measure space (X, d, μ) is a *PI space* if (X, d) is complete, μ is a doubling measure on X and (X, d, μ) satisfies a "$(1, q)$-Poincaré inequality" for some $1 \leq q < \infty$: There is a constant $C > 0$ such that, for every compactly supported Lipschitz function $f \colon X \to \mathbb{R}$ and every open ball B in X,

$$\fint_B |f - f_B| d\mu \leq C(\mathrm{diam}\ B) \left(\fint_{CB} (\mathrm{Lip}_f)^q d\mu \right)^{1/q}.$$

(Here the notations $\fint_E g d\mu$ and g_E both denote the average value of the function g on the set E, i.e., $\frac{1}{\mu(E)} \int_E g d\mu$.)

This definition can be found in [13]; it is equivalent to other versions of the Poincaré inequality in metric measure spaces, such as the original one of [12]. If X is a PI space, then the collection of constants associated to the doubling property and Poincaré inequality on X are known as the *data of* X.

In addition to providing a differentiable structure (see below), the PI space property of X will supply two other key facts for us, summarized in the following proposition.

Proposition 2.4. *Let (X, d, μ) be a PI space. Then there is a constant $C > 0$, depending only on the data of X, such that the following two statements hold:*

(a) *X is quasiconvex, meaning that any two points $x, y \in X$ can be joined by a rectifiable path of length at most $Cd(x,y)$.*
(b) *For any bounded Lipschitz function u on X,*

$$\mathrm{LIP}(u) \leq C \|\mathrm{Lip}_u\|_\infty.$$

Proof. The first statement can be found in Theorem 17.1 of [6]. The second can be found (in greater generality than we need here) in [9], Theorem 4.7. ☐

The following definition is essentially due to Cheeger, in Section 4 of [6]. The form we state can be found in Definition 2.1.1 of [14] (see also [4]). The notation $\langle \cdot, \cdot \rangle$ denotes the standard inner product on Euclidean space of the appropriate dimension.

Definition 2.5. Let (X, d, μ) be a metric measure space. Let $\{U_j\}_{j\in J}$ be a collection of pairwise disjoint measurable sets covering X, let $\{k_j\}_{j\in J}$ be a collection of non-negative integers, and let $\{\phi_j \colon X \to \mathbb{R}^{k_j}\}_{j\in J}$ be a collection of Lipschitz functions.

We say that the collection $\{(U_j, \phi_j)\}$ forms a *measurable differentiable structure* for X if the following holds: For every Lipschitz function u on X and every $j \in J$, there is a Borel measurable function $d^j u \colon U_j \to \mathbb{R}^{k_j}$ such that, for almost every $x \in U_j$,

$$\lim_{y\to x} \frac{|u(y) - u(x) - \langle d^j u(x), (\phi_j(y) - \phi_j(x))\rangle|}{d(y,x)} = 0. \tag{2.6}$$

Furthermore, the function $d^j u$ should be unique (up to sets of measure zero).

We call each pair (U_j, ϕ_j) a *chart* for the differentiable structure on X. For more background on differentiable structures (also called "strong measurable differentiable structures" and "Lipschitz differentiability spaces") see [14, 3].

Note that the defining property (2.6) for a measurable differentiable structure can be more succinctly rephrased as

$$\mathrm{Lip}_{u - \langle d^j u(x), \phi_j\rangle}(x) = 0.$$

The link between PI spaces and measurable differentiable structures is given by the following theorem of Cheeger, one of the main results of [6]. (See also [14, 15, 3] for alternate approaches.)

Theorem 2.7 ([6], Theorem 4.38)**.** *Every PI space X supports a measurable differentiable structure*

$$\{(U_j, \phi_j \colon X \to \mathbb{R}^{k_j})\},$$

and the dimensions k_j of the charts U_j are bounded by a uniform constant depending only on the constants associated to the doubling property and Poincaré inequality of X.

If X supports a measurable differentiable structure, then it generally supports many other equivalent ones. For example, the sets U_j may be decomposed into measurable pieces or the functions ϕ_j rescaled without altering the properties in Definition 2.5. At times, it will be helpful to assume certain extra properties of the charts.

Definition 2.8. A measurable differentiable structure $\{(U_j, \phi_j \colon X \to \mathbb{R}^{k_j})\}_{j\in J}$ is *normalized* if, for each $j \in J$, there exists $c_j > 0$, such that

$$U_j \text{ is closed,} \tag{2.9}$$

$$\mathrm{LIP}(\phi_j) = 1, \text{ and} \tag{2.10}$$

$$|d^j u(x)| \leq c_j \mathrm{Lip}_u(x) \text{ whenever } u \text{ is differentiable at } x \in U_j. \tag{2.11}$$

The definition of a normalized chart is a minor modification of the notion of a "structured chart", due to Bate ([3], Definition 3.6). The following lemma, essentially due to Bate, says that a given chart structure on X can always be normalized by rescaling and chopping.

Lemma 2.12. *Let X be a PI space and let $\{(U_j, \phi_j : X \to \mathbb{R}^{k_j})\}_{j\in J}$ be a measurable differentiable structure on X. Then there exists a collection of sets $\{U_{j,k}\}_{j\in J, k\in K_j}$ such that*

- *each set $U_{j,k}$ is contained in U_j and*
- *$\{(U_{j,k}, (\mathrm{LIP}(\phi_j))^{-1}\phi_j)\}_{j\in J,k\in K_j}$ is a normalized measurable differentiable structure on X.*

Proof. By Lemma 3.4 of [3], we can decompose each chart U_j into charts $U_{j,k}$ such that the measurable differentiable structure $\{(U_{j,k}, \phi_j)\}$ satisfies (2.11).

As $(\mathrm{LIP}(\phi_j))^{-1}\phi_j)$ is just a rescaling of ϕ_j, the chart $(U_{j,k}, (\mathrm{LIP}(\phi_j))^{-1}\phi_j)$ still possesses property (2.11) (with a different constant $c_{j,k}$).

As a final step, we decompose each $U_{j,k}$ into closed sets, up to measure zero, while maintaining the same chart functions. □

For technical reasons, it will be convenient in the proofs of Theorems 1.3 and 1.8 that the measurable differentiable structure is normalized. That this can be done without loss of generality is the content of the following simple lemma.

Lemma 2.13. *To prove Theorems 1.3 and 1.8, we can assume without loss of generality that the measurable differentiable structure $\{(U_j, \phi_j)\}$ is normalized.*

Proof. Assume that we can prove Theorems 1.3 and 1.8 if the charts involved are normalized.

Suppose (U_j, ϕ_j) is an arbitrary (not necessarily normalized) measurable differentiable structure on X. Let $\Omega \subset X$ be open with $\mu(\Omega) < \infty$, let $\{f_j : U_j \cap \Omega \to \mathbb{R}^{k_j}\}_{j\in J}$ be a collection of Borel functions, and let $\epsilon > 0$.

By Lemma 2.12, there is a normalized measurable differentiable structure

$$\{(U_{j,k}, (\mathrm{LIP}(\phi_j))^{-1}\phi_j)\}_{j\in J,k\in K_j} \tag{2.14}$$

on X, where each $U_{j,k}$ is contained in U_j.

Let $g_{j,k} = (\mathrm{LIP}\phi_j)f_j$. Apply Theorem 1.3 to the normalized measurable differentiable structure (2.14), with the functions $g_{j,k}$ and the same parameter ϵ. We immediately obtain an open set $A \subset \Omega$ and a Lipschitz function $u \in C_0(\Omega)$ that satisfy all four requirements of Theorem 1.3.

A similar argument applies to reduce Theorem 1.8 to the normalized case. □

The original arguments of [1] and [18] to prove Theorems 1.1 and 1.2 use the dyadic cube decomposition of Euclidean space. We will use the analogous decomposition in arbitrary doubling metric spaces provided by a result of Christ [8].

Proposition 2.15 ([8], Theorem 11)**.** *Let (X, d, μ) be a doubling metric measure space. Then there exist constants $c \in (0, 1)$, $\eta > 0$, $a_0 > 0$, $a_1 > 0$, and $C_1 > 0$ such that for each $k \in \mathbb{Z}$ there is a collection $\Delta_k = \{Q_i^k : i \in I_k\}$ of disjoint open subsets of X with the following properties:*

(i) *For each $k \in \mathbb{Z}$, $\mu(X \setminus \cup_i Q_i^k) = 0$.*

(ii) *For each $k \in \mathbb{Z}$ and $i \in I_k$, there is a point $z_i^k \in Q_i^k$ such that*
$$B(z_i^k, a_0 c^k) \subset Q_i^k \subset B(z_i^k, a_1 c^k).$$

(iii) *For each $k \in \mathbb{Z}$ and $i \in I_k$, and for each $t > 0$,*
$$\mu\left(\{x \in Q_i^k : \mathrm{dist}(x, X \setminus Q_i^k) \le tc^k\}\right) \le C_1 t^\eta \mu(Q_i^k).$$

(iv) *If $\ell \ge k$, $Q \in \Delta_\ell$, and $Q' \in \Delta_k$, then either $Q \subset Q'$ or $Q \cap Q' = \emptyset$.*

(v) *For each $k \in \mathbb{Z}$, each $i \in I_k$, and each integer $\ell < k$ there is a unique $j \in I_\ell$ such that $Q_i^k \subset Q_j^\ell$.*

We refer to the elements of any Δ_k as cubes.

The next lemma is one of the primary differences between our proof of Theorem 1.3 and the proof of Theorem 1.1 from [1]. It allows us to replace a single-scale argument in [1] by an argument that uses multiple scales simultaneously, which will allow us to deal with the presence of multiple charts.

Lemma 2.16. *Let (X, d, μ) be a complete doubling metric measure space. Suppose that $\mu(X \setminus \cup_{j\in J} U_j) = 0$, where the sets U_j are disjoint and measurable. Fix $\gamma > 0$ and positive numbers $\{\delta_j\}_{j\in J}$. Then we can find a collection $\mathcal{T}$ of pairwise disjoint cubes in X (of possibly different scales) such that the following conditions hold:*

(i) $\mu(X \setminus \bigcup_{T\in\mathcal{T}} T) = 0$.

(ii) There is a map $j\colon \mathcal{T} \to J$ such that

$$\mu(U_{j(T)} \cap T) \geq (1-\gamma)\mu(T) \tag{2.17}$$

and

$$\operatorname{diam} T < \delta_{j(T)} \tag{2.18}$$

for each $T \in \mathcal{T}$.

Proof. Let us call a cube $T \in \Delta_k$ "good for j" if it satisfies (2.17) and (2.18) with $j(T) = j$, and let us call T "good" if it is good for some $j \in J$. Finally, let us call T "bad" if it is not good. Write Δ_k^g for the sub-collection of Δ_k consisting of good cubes.

We then define our collection of cubes $\mathcal{T}$ to be

$$\mathcal{T} = \bigcup_{k=1}^{\infty} \{T \in \Delta_k^g : \text{for every } 1 \leq k' < k \text{ and every } Q \in \Delta_{k'} \text{ containing } T,\ Q \text{ is bad}\}.$$

In other words, our collection consists of all cubes that are the first good cube among all their ancestors of scales below 1. Note that any two distinct cubes in $\mathcal{T}$ are disjoint: if not, then one would contain the other, forcing the larger one to be bad.

For each cube T in this collection $\mathcal{T}$, define $j(T)$ to be a choice of $j \in J$ such that T is good for j. The collection $\mathcal{T}$ and the map $j : \mathcal{T} \to J$ then automatically satisfy condition (ii) of the Lemma.

To verify (i), we show that almost every point $x \in X$ is contained in one of the cubes $T \in \mathcal{T}$. Let

$$Z = X \setminus \bigcap_{k\in\mathbb{Z}} \bigcup_{\ell\in I_k} Q_\ell^k,$$

so that $\mu(Z) = 0$ by Proposition 2.15 (i).

Let $x \in X \setminus Z$ be a point of μ-density of some U_{j_0}. We claim that $x \in T$ for some $T \in \mathcal{T}$. Suppose, to the contrary, that $x \notin T$ for any $T \in \mathcal{T}$. Then x lies in an infinite nested sequence of bad cubes. But this is impossible: if an infinite nested sequence of cubes satisfied $Q_1 \supset Q_2 \supset \cdots \ni x$, then eventually some Q_i would be good for j_0, and the first such good cube would be in $\mathcal{T}$.

So $\cup_{T\in\mathcal{T}} T$ contains almost every point in $(\cup_{j\in J} U_j) \cap (X \setminus Z)$, which is almost every point of X. □

The following lemma will ensure that we obtain a Lipschitz function in our construction. Recall the definition of quasiconvexity from Proposition 2.4.

Lemma 2.19. *Let X be a complete and quasiconvex metric space and let $u\colon X \to \mathbb{R}$ be a function on X. Suppose that there are pairwise disjoint open sets $A_i \subset X$ $(i \in I)$, and a constant $L \geq 0$ such that*

$$\operatorname{LIP}(u|_{A_i}) \leq L \textit{ for each } i \in I \tag{2.20}$$

and

$$u = 0 \textit{ on } B = X \setminus \cup_{i\in I} A_i. \tag{2.21}$$

Then u is $2CL$-Lipschitz on X, where C is the quasiconvexity constant of X.

Proof. Without loss of generality, we may assume that $A_i \neq X$ for each $i \in I$, otherwise the lemma is trivial. Note also that the assumption (2.20) improves immediately to

$$\operatorname{LIP}(u|_{\overline{A_i}}) \leq L \text{ for each } i \in I.$$

Fix points $x, y \in X$. We will show that

$$|u(x) - u(y)| \leq 2CLd(x, y), \tag{2.22}$$

where C is the quasiconvexity constant of X.

Using the quasiconvexity of X, choose a rectifiable path $\gamma\colon [0,1] \to X$ of length at most $Cd(x,y)$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Case 1: Suppose that, for some $i, j \in I$, we have $x \in A_i$ and $y \in A_j$. In this case, we may also suppose that $i \neq j$, otherwise (2.22) follows from the assumption (2.20). Let $t_0 = \inf\{t : \gamma(t) \notin A_i\}$ and $t_1 = \sup\{t : \gamma(t) \notin A_j\}$. By basic topology, $\gamma(t_0) \in \partial A_i \subset B$ and $\gamma(t_1) \in \partial A_j \subset B$. Thus, we have

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u(\gamma(t_0))| + |u(\gamma(t_0)) - u(\gamma(t_1))| + |u(\gamma(t_1)) - u(y)| \\
&\leq Ld(x, \gamma(t_0)) + 0 + Ld(y, \gamma(t_1)) \\
&\leq (2L)\mathrm{length}(\gamma) \\
&\leq 2CLd(x, y)
\end{aligned}$$

Case 2: Suppose that $x \in A_i$ for some $i \in I$ and that $y \in B$ (or vice versa). We then have that $u(y) = 0$ and

$$d(x, y) \geq \frac{1}{C}\mathrm{length}(\gamma) \geq \frac{1}{C}\mathrm{dist}(x, \partial A_i).$$

Thus,

$$|u(x) - u(y)| = |u(x)| \leq L\mathrm{dist}(x, \partial A_i) \leq CLd(x, y).$$

Case 3: Suppose that $x \in B$ and $y \in B$. Then $u(x) = u(y) = 0$. □

Note that Lemma 2.19 is false without assumption (2.21), as the Cantor staircase function shows.

The following lemma is due to Francos ([10], Lemma 2.3). Although Francos stated it only for subsets of $\mathbb{R}^n$, the proof works equally well in our setting.

Lemma 2.23. *Let X be a PI space and let f be a Borel function from an open set $\Omega \subset X$, with $\mu(\Omega) < \infty$, into some $\mathbb{R}^N$. Then, for any $\epsilon > 0$, there is a compact set $K \subset \Omega$ and a continuous function g on Ω such that*

(a) $\mu(U \setminus K) < \epsilon$,
(b) $f = g$ *on* K,
(c) $\int_\Omega |g|^p \leq 2 \int_\Omega |f|^p$ *for all* $p \in [1, \infty)$, *and*
(d) $\|g\|_\infty \leq 2\|f\|_\infty$.

3. Proof of Theorem 1.3

Our main lemma is the analog of Lemma 7 of [1]:

Lemma 3.1. *Let (X, d, μ) be a PI space and let $\{(U_j, \phi_j\colon X \to \mathbb{R}^{k_j})\}_{j \in J}$ be a normalized measurable differentiable structure on X. Suppose that $\Omega \subset X$ is open with $\mu(\Omega) < \infty$ and $\Omega \neq X$, and that $\{f_j\colon \Omega \to \mathbb{R}^{k_j}\}$ is a uniformly bounded collection of continuous functions, i.e., that $\sup_{j \in J} \|f_j\|_\infty < \infty$. Fix $\alpha, \epsilon > 0$.*

Then there exists a compact set $K \subset \Omega$ and a Lipschitz function $u \in C_c(\Omega)$ such that the following conditions hold:

$$\mu(\Omega \setminus K) \leq \epsilon \mu(\Omega). \tag{3.2}$$

$$|f_j - d^j u| \leq \alpha \text{ a.e. on } U_j \cap K. \tag{3.3}$$

$$\|\mathrm{Lip}_u\|_p \leq C' \epsilon^{\frac{1}{p} - \frac{1}{\eta}} \left(\sum_{j \in J} \int_{U_j \cap \Omega} |f_j|^p \right)^{1/p} \quad \text{for all } p \in [1, \infty). \tag{3.4}$$

$$\|\mathrm{Lip}_u\|_\infty \leq C'\epsilon^{-\frac{1}{\eta}} \sup_{j\in J} \|f_j\|_\infty. \tag{3.5}$$

The constants $\eta, C' > 0$ depend only on the data of X.

Proof. Without loss of generality, we assume that $\epsilon < 1$.

Fix a compact set $K' \subset \Omega$ such that $\mu(\Omega \setminus K') < \frac{\epsilon}{4}\mu(\Omega)$. For each $j \in J$, choose $\delta_j > 0$ small enough such that

$$\text{if } |x-y| < \delta_j \text{ and } x \in K', \text{ then } |f_j(x) - f_j(y)| < \alpha/2, \tag{3.6}$$

and

$$\delta_j < \mathrm{dist}(K', X \setminus \Omega). \tag{3.7}$$

Using Lemma 2.16, we find a collection $\mathcal{T}$ of pairwise disjoint cubes covering almost all of X, and a map $j\colon \mathcal{T} \to J$ such that

$$\mu(U_{j(T)} \cap T) \geq \left(1 - \frac{\epsilon}{4}\right)\mu(T), \tag{3.8}$$

and

$$\mathrm{diam}\ (T) < \delta_{j(T)}. \tag{3.9}$$

for each $T \in \mathcal{T}$.

Consider the sub-collection consisting of all cubes $T \in \mathcal{T}$ such that $T \cap K' \neq \emptyset$. Index these cubes $\{T_i\}_{i\in I}$, and write $j(i)$ for $j(T_i)$. By (3.7) and (3.9), each cube T_i ($i \in I$) lies in Ω.

For each $i \in I$, define $S_i \subset T_i$ as

$$S_i = \{x \in T_i : \mathrm{dist}(x, X \setminus T_i) \geq tc^k\},$$

where k is such that $T \in \Delta_k$ and

$$t = (\epsilon/4C_1)^{1/\eta}$$

is fixed. This value of t was chosen to ensure (by Proposition 2.15 (iii)) that

$$\mu(T_i \setminus S_i) \leq C_1 t^\eta \mu(T_i) = \frac{\epsilon}{4}\mu(T_i).$$

Note that S_i is a compact subset of the open set T_i. Let z_i be a "center" of T_i as in Proposition 2.15 (ii), so that T_i both contains and is contained in a ball centered at z_i of radius approximately diam T_i.

For each cube T_i in our collection, define $a_i \in \mathbb{R}^{k_{j(i)}}$ by

$$a_i = \mu(U_{j(i)} \cap T_i)^{-1} \int_{U_{j(i)}\cap T_i} f_{j(i)}.$$

Note that the collection $\{|a_i|\}_{i\in I}$ is bounded, because the collection $\{f_j\}_{j\in J}$ is uniformly bounded.

Let $\psi_i\colon X \to \mathbb{R}_+$ be a Lipschitz function such that $\psi_i = 1$ on S_i, $\psi_i = 0$ off T_i, and $\mathrm{LIP}(\psi_i) \leq C(tc^k)^{-1} \leq C(\mathrm{diam}\ T_i)^{-1}\epsilon^{-1/\eta}$. (Here C is some constant depending only on the data of X.) By slightly widening the regions where ψ is constant, we can also easily arrange that $\mathrm{Lip}_{\psi_i} = 0$ everywhere in S_i and in $X \setminus T_i$.

Define $u\colon X \to \mathbb{R}$ by

$$u(x) = \sum_{i\in I} \psi_i(x)\langle a_i, \phi_{j(i)}(x) - \phi_{j(i)}(z_i)\rangle. \tag{3.10}$$

A simple calculation shows that, for each $i \in I$,

$$\mathrm{LIP}(u|_{T_i}) \leq C\epsilon^{-1/\eta}|a_i| \leq C\epsilon^{-1/\eta}\sup_i |a_i| < \infty.$$

(Here we used the assumption that the measurable differentiable structure is normalized, and therefore $\mathrm{LIP}(\phi_j) \leq 1$ for each $j \in J$.)

Thus, as $u = 0$ outside $\bigcup_{i\in I} T_i$, we see that u is Lipschitz on X by Lemma 2.19. In addition, $u \in C_c(\Omega)$, with supp $u \subset \overline{\bigcup_{i\in I} T_i} \subset \Omega$, and $d^{j(i)}u = a_i$ a.e. on $S_i \cap U_{j(i)}$.

Let $K_1 = \cup_{i\in I}(S_i \cap U_{j(i)})$, and let K be a compact subset of K_1 such that

$$\mu(K_1 \setminus K) \le \frac{\epsilon}{4}\mu(\Omega).$$

To verify (3.2), note that

$$\mu(T_i \setminus (S_i \cap U_{j(i)})) \le \mu(T_i \setminus S_i) + \mu(T_i \setminus U_{j(i)}) \le \frac{\epsilon}{4}\mu(T_i) + \frac{\epsilon}{4}\mu(T_i) \le \frac{\epsilon}{2}\mu(T_i)$$

for each $i \in I$. Therefore,

$$\mu(\Omega \setminus K_1) \le \mu(\Omega \setminus K') + \sum_i \mu(T_i \setminus (S_i \cap U_{j(i)})) \le \frac{3\epsilon}{4}\mu(\Omega),$$

and so

$$\mu(\Omega \setminus K) \le \mu(\Omega \setminus K_1) + \mu(K_1 \setminus K) \le \epsilon\mu(\Omega).$$

Let us now verify (3.3). Suppose that $x \in U_j \cap K$ for some $j \in J$. Then $x \in S_i \cap U_j$ for some $i \in I$ such that $j(i) = j$. Therefore, by (3.6), $|f_j(x) - a_i| < \alpha$. So $|f_j - a_i| < \alpha$ on $S_i \cap U_j$. Now, since $d^j u = a_i$ almost everywhere in $U_j \cap S_i$, we see that

$$|f_j - d^j u| < \alpha$$

almost everywhere in $U_j \cap S_i$. This verifies (3.3), as $K \subset \cup_i(S_i \cap U_{j(i)})$.

Finally, we must check (3.4) and (3.5). Observe that if T is a cube in $\mathcal{T}$ and $x \in T$, then the sum (3.10) defining u consists of at most one non-zero term. Therefore, for such x, we have by (2.2) that

$$\begin{aligned}\mathrm{Lip}_u(x) &\le \sup_{i\in I}\left(\mathrm{Lip}_{\psi_i}(x)|\langle a_i, \phi_{j(i)}(x) - \phi_{j(i)}(z_i)\rangle| + |a_i|\psi_i(x)\right)\\ &\le \sum_{i\in I}\mathrm{Lip}_{\psi_i}(x)|\langle a_i, \phi_{j(i)}(x) - \phi_{j(i)}(z_i)\rangle| + \sum_{i\in I}|a_i|\psi_i(x).\end{aligned} \tag{3.11}$$

Because almost every $x \in X$ is contained in some $T \in \mathcal{T}$, we have the bound (3.11) for almost every $x \in \Omega$.

Recalling our normalization that $\mathrm{LIP}(\phi_j) \le 1$ for all $j \in J$, we see from (3.11) that, for all $1 \le p < \infty$,

$$\begin{aligned}\|\mathrm{Lip}_u\|_p &\le \left(\sum_{i\in I}\left(\|\mathrm{Lip}_{\psi_i}\|_\infty|a_i|(\mathrm{diam}\ T_i)\right)^p\mu(T_i\setminus S_i)\right)^{1/p} + \left(\sum_{i\in I}|a_i|^p\mu(T_i)\right)^{1/p}\\ &\le \left(\sum_{i\in I}\left(C\epsilon^{-1/\eta}|a_i|\right)^p\epsilon\mu(T_i)\right)^{1/p} + \left(\sum_{i\in I}|a_i|^p\mu(T_i)\right)^{1/p}\\ &\le \left(C\epsilon^{\frac{1}{p}-\frac{1}{\eta}}+1\right)\left(\sum_{i\in I}|a_i|^p\mu(T_i)\right)^{1/p}\\ &\le \left(C\epsilon^{\frac{1}{p}-\frac{1}{\eta}}+1\right)\left(\sum_{i\in I}\frac{\mu(T_i)}{\mu(T_i\cap U_{j(i)})}\int_{T_i\cap U_{j(i)}}|f_{j(i)}|^p\right)^{1/p}\\ &\le 2\left(C\epsilon^{\frac{1}{p}-\frac{1}{\eta}}+1\right)\left(\sum_{j\in J}\int_{\Omega\cap U_j}|f_j|^p\right)^{1/p}\end{aligned}$$

Note that in the last inequality we used (3.8).

The case $p = \infty$, namely (3.5), follows from this by a limiting argument, or can alternatively be derived the same way. This completes the proof of Lemma 3.1. □

Proof of Theorem 1.3. By Lemma 2.13, we may assume that the measurable differentiable structure is normalized.

It will also be convenient to assume that Ω is a proper subset of X, i.e., that $\Omega \neq X$. We may assume this without loss of generality: If $\Omega = X$, we replace Ω by $\Omega' = X \setminus \{x_0\}$ for some arbitrary $x_0 \in X$. Proving Theorem 1.3 for Ω' also proves it for Ω.

Finally, we may also assume that $\epsilon < 1$ and that $\sup_{j\in J} \|f_j\|_\infty > 0$. We also extend each f_j from $U_j \cap \Omega$ to all of Ω by setting $f_j = 0$ off of U_j. The proof now proceeds in two steps.

Step 1: Assume that the functions f_j are uniformly bounded, i.e., that $\sup_{j\in J} \|f_j\|_\infty < \infty$.

Let $\{\alpha_n\}_{n\geq 1}$ be a decreasing sequence of positive real numbers with $\alpha_1 \leq \sup_{j\in J} \|f_j\|_\infty$, to be chosen later. For each integer $n \geq 0$, we will inductively build:

$$\text{a Lipschitz function } u_n \in C_c(\Omega, \mathbb{R}),$$
$$\text{a compact set } K_n \subset \Omega, \text{ and}$$
$$\text{a collection of continuous functions } \{f_j^n \colon \Omega \to \mathbb{R}^{k_j}\}_{j\in J}.$$

Let $u_0 = 0$. For each $j \in J$, we apply Lemma 2.23, to find a compact set $K_{0,j} \subset \Omega$ with $\mu(\Omega \setminus K_{0,j}) < 2^{-1}2^{-j}\epsilon\mu(\Omega)$, and a continuous function f_j^0 on Ω such that $f_j^0 = f_j$ on $K_{0,j}$,

$$\int_\Omega |f_j^0|^p \leq 2\int_\Omega |f_j|^p = 2\int_{\Omega\cap U_j} |f_j|^p \tag{3.12}$$

for all $1 \leq p < \infty$, and

$$\sup |f_j^0| \leq 2\|f_j\|_\infty. \tag{3.13}$$

Let $K_0 = \cap_{j\in J} K_{0,j}$. This completes stage $n = 0$ of the construction.

Suppose now that $u_{n-1}, K_{n-1}, \{f_j^{n-1}\}_{j\in J}$ have been constructed. Apply Lemma 3.1 to get a compact set $\tilde{K}_n \subset \Omega$ and a Lipschitz function $u_n \in C_c(\Omega)$ such that

$$\mu(\Omega \setminus \tilde{K}_n) \leq 2^{-(n+2)}\epsilon\mu(\Omega)$$

,

$$|f_j^{n-1}(x) - (d^j u_n)(x)| \leq \alpha_n/2$$

for every $j \in J$ and almost every $x \in U_j \cap \tilde{K}_n$,

$$\|\mathrm{Lip}_{u_n}\|_p \leq C'(2^{-(n+2)}\epsilon)^{\frac{1}{p}-\frac{1}{\eta}} \left(\sum_j \int_{U_j\cap\Omega} |f_j^{n-1}|^p\right)^{1/p}$$

for every $p \in [1, \infty)$, and

$$\|\mathrm{Lip}_{u_n}\|_\infty \leq C'(2^{-(n+2)}\epsilon)^{-\frac{1}{\eta}} \sup_{j\in J} \|f_j^{n-1}\|_\infty.$$

Given $j \in J$, define $\tilde{f}_j^n(x) \colon \Omega \to \mathbb{R}^{k_j}$ by

$$\tilde{f}_j^n(x) = \begin{cases} f_j^{n-1}(x) - d^j u_n(x) & \text{if } x \in U_j \cap \tilde{K}_n \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

For each $j \in J$, apply Lemma 2.23 to find a compact set $K_{n,j} \subset \Omega$ and a continuous map $f_j^n \colon \Omega \to \mathbb{R}^{k_j}$ such that

$$\mu(\Omega \setminus K_{n,j}) \leq 2^{-(n+2)}2^{-j}\epsilon\mu(\Omega), \tag{3.15}$$

$$f_j^n = \tilde{f}_j^n \text{ on } K_{n,j}, \text{ and} \tag{3.16}$$

$$\|f_j^n\|_\infty \leq 2\|\tilde{f}_j^n\|_\infty \leq \alpha_n. \tag{3.17}$$

Let $K_n = \tilde{K}_n \cap \left(\bigcap_{j\in J} K_{n,j}\right)$, so that

$$\mu(\Omega \setminus K_n) \leq \mu(\tilde{K}_n) + \sum_{j\in J} \mu(\Omega \setminus K_{n,j}) \leq 2^{-(n+1)}\epsilon\mu(\Omega).$$

This completes stage n of the inductive construction.

Now let

$$A = \Omega \setminus \cap_{n=0}^{\infty} K_n = \cup_{n=0}^{\infty}(\Omega \setminus K_n)$$

and

$$u = \sum_{n=0}^{\infty} u_n.$$

Note that

$$\mu(A) \leq \sum_{n=0}^{\infty} \mu(\Omega \setminus K_n) \leq \epsilon\mu(\Omega),$$

so (1.4) holds.

Purely for notational convenience, we now define a real-valued function

$$F = \sum_{j\in J} \chi_{U_j} |f_j|,$$

so that

$$\|F\|_p = \left(\sum_j \int_{U_j\cap\Omega} |f_j|^p\right)^{1/p}$$

for every $p \in [1, \infty)$ and

$$\|F\|_\infty = \sup_{j\in J} \|f_j\|_\infty.$$

Note that, if $p \in [1, \infty)$, we have, by (3.12) and (3.13), that

$$\left(\sum_{j\in J} \int_{U_j\cap\Omega} |f_j^0|^p\right)^{1/p} \leq 2\|F\|_p \qquad \text{and} \qquad \sup_{j\in J} \|f_j^0\|_\infty \leq 2\|F\|_\infty$$

In addition, $\|F\|_p$ is non-zero and finite for every $p \in [1, \infty]$, by our assumption that $0 < \sup_{j\in J} \|f_j\|_\infty < \infty$.

We now calculate that, for $p \in [1,\infty)$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \|\mathrm{Lip}_{u_n}\|_p &\leq \sum_{n=1}^{\infty} C' 2^{\frac{n+2}{\eta}} \epsilon^{\frac{1}{p}-\frac{1}{\eta}} \left(\sum_{j\in J} \int_{U_j\cap\Omega} |f_j^{n-1}|^p \right)^{1/p}\\
&\leq 2C'\epsilon^{\frac{1}{p}-\frac{1}{\eta}} \left(\|F\|_p + \sum_{n=1}^{\infty} 2^{\frac{n+2}{\eta}} \left(\sum_{j\in J} \int_{U_j\cap\Omega} |f_j^{n}|^p \right)^{1/p} \right)\\
&\leq 2C'\epsilon^{\frac{1}{p}-\frac{1}{\eta}} \left(\|F\|_p + \sum_{n=1}^{\infty} 2^{\frac{n+2}{\eta}} \left(\sum_{j\in J} \|f_j^n\|_\infty^p \mu(U_j\cap\Omega) \right)^{1/p} \right)\\
&\leq 2C'\epsilon^{\frac{1}{p}-\frac{1}{\eta}} \left(\|F\|_p + \sum_{n=1}^{\infty} 2^{\frac{n+2}{\eta}} \alpha_n \mu(\Omega)^{1/p} \right)\\
&\leq 2C'\epsilon^{\frac{1}{p}-\frac{1}{\eta}} \|F\|_p \left(1 + \frac{\mu(\Omega)^{1/p}}{\|F\|_p} \sum_{n=1}^{\infty} 2^{\frac{n+2}{\eta}} \alpha_n \right). \qquad (3.18)
\end{aligned}$$

A similar calculation shows that (3.18) also holds if $p=\infty$.

The function $p \mapsto \frac{\mu(\Omega)^{1/p}}{\|F\|_p}$ is continuous for $p\in[1,\infty]$, and therefore has an upper bound $M>0$. Choose our sequence $\{\alpha_n\}$ to satisfy

$$\sum 2^{\frac{n+2}{\eta}} \alpha_n \leq 1/M.$$

Then the calculation (3.18) yields that

$$\sum_{n=1}^{\infty} \|\mathrm{Lip}_{u_n}\|_p \leq 4C'\epsilon^{\frac{1}{p}-\frac{1}{\eta}} \|F\|_p < \infty \qquad (3.19)$$

for any $p\in[1,\infty]$.

Proposition 2.4 says that, for each n,

$$\mathrm{LIP}(u_n) \leq C\|\mathrm{Lip}_{u_n}\|_\infty,$$

and therefore (3.19) implies that

$$\sum_{n=1}^{\infty} \mathrm{LIP}(u_n) < \infty. \qquad (3.20)$$

This, combined with the fact that each u_n has compact support in $\Omega \neq X$, implies that the sum

$$u = \sum_{n=1}^{\infty} u_n$$

converges uniformly on compact sets to a Lipschitz function $u \in C_0(\Omega)$. It follows from (3.20), (3.19), and (2.1) that u satisfies conditions (1.6) and (1.7) of Theorem 1.3.

To verify (1.5), fix $j\in J$ and observe that, by (3.14) and (3.16),

$$f_j - \sum_{n=1}^{m} (d^j u_n) = f_j^m,$$

almost everywhere in $U_j \cap (\Omega \setminus A)$, for each positive integer m. Thus,

$$\begin{aligned}\|f_j - d^j u\|_{L^\infty(U_j\cap(\Omega\setminus A))} &\le \|f_j^m\|_{L^\infty(U_j\cap(\Omega\setminus A))} + \sum_{n=m+1}^{\infty} \|d^j u_n\|_{L^\infty(U_j\cap(\Omega\setminus A))} \\ &\le \alpha_m + \sum_{n=m+1}^{\infty} \|d^j u_n\|_{L^\infty(U_j\cap(\Omega\setminus A))} \\ &\le \alpha_m + c_j \sum_{n=m+1}^{\infty} \|\mathrm{Lip}_{u_n}\|_{L^\infty(U_j\cap(\Omega\setminus A))}\end{aligned}$$

and both of these tend to zero as m tends to infinity. In the last inequality, we used the fact that (U_j, ϕ_j) is a normalized chart, see Definition 2.8.

Thus,

$$f_j = d^j u \text{ a.e. on } U_j \cap (\Omega \setminus A),$$

so (1.5) holds. This completes Step 1.

Step 2: The functions $\{f_j \colon U_j \cap \Omega \to \mathbb{R}^{k_j}\}$ are arbitrary Borel functions.

We first extend each f_j to be zero off of U_j, so that each f_j is defined on all of Ω.

Fix $\epsilon > 0$. Choose $r > 0$ large so that

$$B = \{x : |f_j(x)| > r \text{ for some } j \in J\}$$

satisfies $\mu(B) < \epsilon/2$. Note that this is possible because, using the fact that $f_j = 0$ off U_j, we see that

$$\mu\left(\Omega \setminus \bigcup_{\ell=1}^{\infty} \{x : f_j(x) \le \ell \text{ for all } j \in J\}\right) = 0.$$

For each $j \in J$ let

$$\tilde{f}_j(x) = \begin{cases} f_j(x) & \text{if } |f_j(x)| \le r \\ r f_j(x)/|f_j(x)| & \text{if } |f_j(x)| > r. \end{cases}$$

Then $\{\tilde{f}_j\}$ is a uniformly bounded collection of Borel functions on Ω such that, for all $j \in J$, $|\tilde{f}_j| \le |f_j|$ everywhere and $\tilde{f}_j = f_j$ outside the set B. Fix an open set $A_1 \supseteq B$ such that $\mu(A_1) < \epsilon/2$. Then, for all $j \in J$, $\tilde{f}_j = f_j$ outside of A_1.

Now apply the result of Step 1 to the uniformly bounded collection $\{\tilde{f}_j\}$. We obtain an open set A_2 with $\mu(A_2) \le \frac{\epsilon}{2}\mu(\Omega)$ and a Lipschitz function $u \in C_0(\Omega)$ such that

$$d^j u = \tilde{f}_j \text{ a.e. in } U_j \cap (\Omega \setminus A_2),$$

$$\|\mathrm{Lip}_u\|_p \le 4C'(\epsilon/2)^{\frac{1}{p}-\frac{1}{\eta}} \left(\sum_{j\in J} \int_{U_j\cap\Omega} |\tilde{f}_j|^p\right)^{1/p}$$

for all $p \in [1, \infty)$, and

$$\|\mathrm{Lip}_u\|_\infty \le 4C'(\epsilon/2)^{-\frac{1}{\eta}} \sup_{j\in J} |\tilde{f}_j|_\infty.$$

Thus, for each $j \in J$, $f_j = d^j u$ a.e. in $U_j \cap (\Omega \setminus A)$, where $A = A_1 \cup A_2$ has $\mu(A) \le \epsilon\mu(\Omega)$. This verifies (1.4) and (1.5).

If $p \in [1, \infty)$, we have

$$\|\mathrm{Lip}_u\|_p \le 4C'(\epsilon/2)^{\frac{1}{p}-\frac{1}{\eta}} \left(\sum_{j\in J} \int_{U_j\cap\Omega} |\tilde{f}_j|^p\right)^{1/p} \le 4C' 2^{1/\eta} \epsilon^{\frac{1}{p}-\frac{1}{\eta}} \left(\sum_{j\in J} \int_{U_j\cap\Omega} |f_j|^p\right)^{1/p},$$

which verifies (1.6). A similar calculation verifies (1.7).

This completes the proof of Theorem 1.3. □

4. Proof of Theorem 1.8

In this section, we give the proof of Theorem 1.8. Given our Theorem 1.3, we can now just closely follow the proof given by Moonens-Pfeffer in [18]. For the convenience of the reader, we give most of the details, although they are very similar to those of [18].

In our setting, the analog of Corollary 1.2 in [18] is the following:

Lemma 4.1. *Let X be a PI space with a normalized differentiable structure $(U_j, \phi_j : U_j \to \mathbb{R}^{k_j})$. Let $\Omega \subset X$ be a bounded open subset of X and let $\{f_j : U_j \cap \Omega \to \mathbb{R}^{k_j}\}$ be a collection of Borel functions. Then for every $\epsilon > 0$, there exist a compact set $K \subset U$ and a Lipschitz function $u \in C_c(\Omega)$ such that*

$$\mu(\Omega \setminus K) < \epsilon, \tag{4.2}$$

$$d^j u = f_j \text{ a.e. in } U_j \cap K \tag{4.3}$$

for each $j \in J$, and

$$|u(x)| \leq \epsilon \min\{1, \operatorname{dist}^2(x, X \setminus \Omega)\} \tag{4.4}$$

for all $x \in X$.

Proof. We can again assume without loss of generality that $\Omega \neq X$, otherwise we replace $\Omega = X$ by $X \setminus \{x_0\}$ for some $x_0 \in X$. Extend the functions f_j to all of X by letting $f_j = 0$ off of $U_j \cap \Omega$.

Fix an open set Ω' compactly contained in Ω with $\mu(\Omega \setminus \Omega') < \epsilon/2$.

As in Step 2 in the proof of Theorem 1.3, we can find a compact set $B \subset \Omega'$ such that $\mu(\Omega' \setminus B) < \epsilon/4$ and $\{f_j\}$ are uniformly bounded on B, i.e., $\sup_{j\in J} \|f_j\|_{L^\infty(B)} = M < \infty$.

For each $j \in J$, let $g_j = f_j \chi_B$, so the functions g_j are uniformly bounded by the constant $M > 0$. Let

$$\Delta = \min\{1, \operatorname{dist}^2(\Omega', X \setminus \Omega)\}$$

and

$$d = \epsilon^{1+\frac{1}{\eta}} \Delta / (1 + (8\mu(\Omega'))^{\frac{1}{\eta}} CM),$$

where C and η are the constants from Theorem 1.3.

Choose k large so that there are cubes $Q_1, \dots, Q_m \subset \Omega'$ in Δ_k, of diameter at most d, that satisfy

$$\mu(\Omega' \setminus \cup_1^m Q_i) < \epsilon/4.$$

(Note that the doubling property of μ and the boundedness of Ω implies that the collection $\{Q_1, \dots, Q_m\}$ really is finite.)

For each $1 \leq i \leq m$, we now apply Theorem 1.3 to the collection $\{g_j\}$ in the cube Q_i with parameter $\epsilon' = \epsilon/8\mu(\Omega')$. For each $1 \leq i \leq m$, we obtain a compact set $K_i \subset Q_i$ with $\mu(Q_i \setminus K_i) \leq \epsilon' \mu(Q_i)$ and a Lipschitz function $u_i \in C_c(Q_i)$ such that, for each $j \in J$, $d^j u_i = g_j$ almost everywhere in $U_j \cap K_i$.

Furthermore, the remark after the statement of Theorem 1.3 shows that

$$\operatorname{LIP} u_i \leq C(\epsilon')^{-\frac{1}{\eta}} M.$$

As $u_i \in C_c(Q_i)$, it follows that, for each $1 \leq i \leq m$,

$$\|u_i\|_\infty \leq (\operatorname{diam} Q_i) \operatorname{LIP}(u_i) \leq dC(\epsilon')^{-\frac{1}{\eta}} M < \epsilon\Delta.$$

Let $K = B \cap (\cup_{i=1}^m K_i)$, a compact subset of Ω. Our choices easily imply that

$$\mu(\Omega \setminus K) < \epsilon,$$

which verifies (4.2).

Let $u = \sum_{i=1}^m u_i$. Then u is a Lipschitz function in $C_c(\Omega)$ that satisfies

$$d^j u = f_j \text{ almost everywhere in } U_j \cap K,$$

so (4.3) holds.

To verify the final condition, note that u is identically zero outside of Ω', so (4.4) holds there automatically. For $x \in \Omega'$, we have

$$|u(x)| \leq \sup_i \|u_i\|_\infty < \epsilon\Delta \leq \epsilon \min\{1, \operatorname{dist}^2(x, X \setminus \Omega)\}.$$

Thus, the final condition (4.4) of Lemma 4.1 is verified. □

We now prove Theorem 1.8. (To avoid some cumbersome subscripts, we change notation slightly and write $\operatorname{Lip}(g)(x)$ instead of $\operatorname{Lip}_g(x)$.)

Proof of Theorem 1.8. We again closely follow [18].

By Lemma 2.13, we may assume that the measurable differentiable structure is normalized. Without loss of generality, we also assume that $\epsilon < 1$. Fix $x_0 \in X$ and let $B_i = B(x_0, i)$ for each $i \in \mathbb{N}$.

We repeatedly apply Lemma 4.1. We inductively construct compact sets $K_i \subset \Omega_i = \Omega \cap B_i \setminus \cup_{k=1}^{i-1} K_k$ and Lipschitz functions $u_i \in C_c(\Omega_i)$ such that, for each $i \in \mathbb{N}$

$$\mu(\Omega_i \setminus K_i) < 2^{-i}\epsilon < 2^{-i}, \tag{4.5}$$

$$d^j u_i = f_j - \sum_{k=1}^{i-1} d^j u_k \text{ a.e. in } K_i, \tag{4.6}$$

and

$$|u_i(x)| \leq 2^{-i}\epsilon \min\{1, \operatorname{dist}^2(x, X \setminus \Omega_i)\} \tag{4.7}$$

for all $x \in X$.

Let $K = \cup_{i=1}^\infty K_i$ and let $u = \sum_{i=1}^\infty u_i$. Note that u is a continuous function, because the bound $\|u_i\|_\infty \leq 2^{-i}\epsilon$ from (4.7) implies the uniform convergence of this sum. It also implies that $\|u\|_\infty \leq \epsilon$, verifying the first part of (1.9).

The second part of (1.9) also follows immediately, by observing that

$$\{u \neq 0\} \subset \bigcup_{i=1}^\infty \{u_i \neq 0\} \subset \bigcup_{i=1}^\infty \Omega_i \subset \Omega.$$

In addition, $\mu\left((\Omega \cap B_i) \setminus K\right) \leq \mu(\Omega_k \setminus K_k) \leq 2^{-k}$ whenever $k \geq i$, which implies that $\mu(\Omega \cap B_i \setminus K) = 0$ for each $i \in \mathbb{N}$ and thus that $\mu(\Omega \setminus K) = 0$.

It remains to verify 1.10 and 1.11.

We first claim that if $x \in K_i$ and $k > i$, then

$$\operatorname{Lip}\left(\sum_{k=i+1}^\infty u_k\right)(x) = 0. \tag{4.8}$$

Indeed, note that for $k > i$ and $x \in K_i$, we have $K_i \cap \Omega_k = \emptyset$ and so $u_k(x) = 0$. Fix any $y \in X$. If $y \notin \Omega_k$, then $u(y) = 0$ as well. If $y \in \Omega_k$, then

$$|u_k(y) - u_k(x)| = |u_k(y)| \leq 2^{-i}\epsilon d(x,y)^2$$

by (4.7). So, in either case, we have

$$|u_k(y) - u_k(x)| \leq 2^{-i}\epsilon d(x,y)^2$$

whenever $x \in K_i$, $y \in X$, and $k > i$. Summing this over all $k > i$ immediately proves (4.8).

Therefore, for almost every $x \in K_i \cap U_j$, we have the following:

$$\begin{aligned}
\mathrm{Lip}(u - f_j(x)\cdot\phi_j)(x) &\leq \mathrm{Lip}\left(\sum_{k=1}^{i} u_k - f_j(x)\cdot\phi_j\right)(x) \\
&\leq \mathrm{Lip}\left(\sum_{k=1}^{i} u_k - \left(\sum_{k=1}^{i} d^j u_k(x)\right)\cdot\phi_j\right)(x) \\
&= 0.
\end{aligned}$$

It follows that at almost every point in $K_i \cap U_j$, the function u is differentiable with $d^j u = f_j$. Because $\mu(\Omega \setminus \cup K_i) = 0$, it follows that $d^j u = f_j$ almost everywhere in $\Omega \cap U_j$. This proves (1.10).

Finally, we must show (1.11), that $\mathrm{Lip}_u = 0$ everywhere in $X \setminus \Omega$. Fix $x \in X \setminus \Omega$. If $y \in X \setminus \Omega$, then $u(x) = u(y) = 0$.

If $y \in \Omega$, then

$$|u(y) - u(x)| = |u(y)| \leq \epsilon \mathrm{dist}^2(x, X\setminus\Omega) \leq \epsilon d(x,y)^2.$$

Thus, for any $x \in X \setminus \Omega$ and any $y \in X$, we have

$$|u(y) - u(x)| \leq \epsilon d(x,y)^2,$$

which immediately implies that $\mathrm{Lip}_u(x) = 0$. This completes the argument. □

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